Exercises

1. Review the proof that $\phi(n)$ is a multiplicative function.
2. Determine whether the following arithmetic functions are are multiplicative? if so, are they completely multiplicative?
   - $f(n) = n!$
   - $f(n) = n^e$ for some $e \in \mathbb{N}$.
   - $f(n) = n/2$
   - $f(n) = \sum_{d|n} d^e$ for some $e \in \mathbb{N}$.
3. Find $\phi(256)$; $\phi(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11)$ and $\phi(2^{e_1} \cdot 3^{e_2} \cdot 5^{e_3} \cdot 7^{e_4} \cdot 11^{e_5})$ for $e_i \in \mathbb{N}$.
4. Is it possible $\phi(n) \geq \phi(n+1)$? If so find an example; otherwise prove it.
5. Find all positive integers n such that $\phi(n) = 6$. (Prove that you have found all possible solutions).
6. Show that if n is a positive integer, then $\phi(2n) = \phi(n)$ if n is odd. Show that if n is a positive integer, then $\phi(2n) = 2\phi(n)$ if n is even. (Hints: factorize $2n$).
7. Prove that there are infinitely many integers n for which $\phi(n)$ is a perfect square. (Hints: consider $n = p^k$.)
8. Prove that for $n \geq 1, k \geq 1$, $\phi(n^k) = n^{k-1}\phi(n)$.
9. Suppose a and n are relatively prime and $k \geq 1$. Prove that $a^{\phi(n^k)} \equiv 1 \pmod{n}$.
10. Let $n = 35$ and $n = 2^5 3^4 5^3 7^3 13$, find $\tau(n)$, $\sigma(n)$ and $\Phi(n)$.
11. Show that $\tau(n)$ is odd iff n is a perfect square.
12. Which positive integers have exactly two positive divisors.
13. Prove that $\tau(n) < 2\sqrt{n}$. (Hints: consider the size of factors of n.)