Exercises

1. Find all integer solutions to the following system of linear congruences,

$$\begin{aligned} x &\equiv 0 \pmod 2 \\ x &\equiv 0 \pmod 3 \\ x &\equiv 2 \pmod 5 \\ x &\equiv 6 \pmod 7 \end{aligned}$$

2. Find all solutions $\pmod{187}$ to the following system of linear congruences,

$$\begin{aligned} x &\equiv 4 \pmod{11} \\ x &\equiv 1 \pmod{17} \end{aligned}$$

3. Find all solutions $\pmod{105}$ to the following system of linear congruences,

$$\begin{aligned} 2x &\equiv 1 \pmod 3 \\ 3x &\equiv 2 \pmod 5 \\ 4x &\equiv 3 \pmod 7 \end{aligned}$$

4. Find an integer that leaves remainders of 2 when divided by 3 and 5, but that is divisible by 4.
5. Find all solutions $\pmod{187}$ to the following system of linear congruences,

$$\begin{aligned} x &\equiv 4 \pmod{11} \\ x &\equiv 4 \pmod{17} \end{aligned}$$

6. If x satisfies the following equations where p_1, p_2, p_3 are pairwisely coprime,

$$\begin{aligned} ax &\equiv b \pmod{p_1} \\ ax &\equiv b \pmod{p_2} \\ ax &\equiv b \pmod{p_3} \end{aligned}$$

   is it true that $ax \equiv b \pmod{p_1 p_2 p_3}$? If so prove it; otherwise find a counterexample.
7. Show that if a is an integer such that a is not divisible by 3 or such that a is divisible by 9, then $a^7 \equiv a \pmod{63}$.
8. Show that if a, b are coprime positive integers, then $a^{\phi(b)} + b^{\phi(a)} \equiv 1 \pmod{ab}$.

   Hints: use CRT and Euler's theorem.
9. Compute $2^{73} \pmod 7$ and $7^{73} \pmod{13}$. Show how to use them to find $72^{73} \pmod{91}$ (where $91 = 7 \cdot 13$).
10. Find $72^{73} \pmod{91}$ directly using Euler's theorem.