Exercises

1. Is $10! + 1$ divisible by 11 and why?
2. What is the remainder when $5! \cdot 25!$ is divided by 31? (Hints: $30 \equiv -1 \pmod{31}$.)
3. Use Wilson's theorem to find the remainder of 149! divided by a prime 139. (Hints: inspection)
4. Show that if p is an odd prime, then $2(p-3)! \equiv -1 \pmod{p}$. Hints: $(p-1)! = (p-2)! \cdot (p-1)$.
5. What is the remainder when 5^{100} is divided by 7?
6. What is the remainder when 41^{75} is divided by 3?
7. Find the last digit of 7^{1000}.
8. What are the last two digits of $3^{3^{333}}$? What are the last two digits of $(3^3)^{333}$? Note that $\phi(100) = 40$ and $\phi(40) = 16$.
9. Let p be a prime and $p \nmid a$. Show that there exists solution x for

   $$ax \equiv b \pmod{p}.$$

   such that $x = a^t b$ for some t. Express such t in terms of p.
10. Let $\lambda(n)$ be the smallest positive integer such that for every integer a relatively prime to n,

    $$a^{\lambda(n)} \equiv 1 \pmod{n}.$$

    The $\lambda(n)$ is named the Carmichael function. Compute $\lambda(n)$ for $n = 7$ and $n = 8$. Is it possible $\lambda(n) < \phi(n)$?