(1) (10 marks) Let $K = \mathbb{Q}(\sqrt{-5})$ and O_K be its ring of integers.

- Show that the norm of the ideal $I = \langle 7, 1+\sqrt{-5}\rangle$ is 1. Then find a basis for I as $\mathbb{Z}$-module.
- Find a $\mathbb{Z}$-module basis for J where $J = \langle 3, 1+\sqrt{-5}\rangle$.
- Compute the norm of the ideal $\langle 7, 1-2\sqrt{-5}\rangle$ by factoring $\langle 7\rangle$.
- Justify why there exists an integral ideal A such that
$$\langle 1+2\sqrt{-5}\rangle = A \cdot \langle 3, 1+2\sqrt{-5}\rangle$$
and then determine A.
- Justify that the fractional ideal $\langle 3, 1+2\sqrt{-5}\rangle^{-1}$ is $\langle 1, \frac{1}{3}+\frac{1}{3}\sqrt{-5}\rangle$. Use this to find the inverse for ideal $\langle 3, 1-2\sqrt{-5}\rangle$.
- Let T be the integral ideal generated by $1+\sqrt{-5}, 3+\sqrt{-5}, 19+9\sqrt{-5}$. Determine α, β such that $T = \langle \alpha, \beta\rangle$.
- Compute the class number of K using Minkowski's bound.
- Let P, Q be two non-principal ideals in K. Justify that PQ^{-1} is a principal ideal.