(1) (6 marks) Let $R = \mathbb{Z} + \mathbb{Z}\sqrt{-5}$. Let I be an ideal generated by 2 and $1 + \sqrt{-5}$.

- Is I a free R-module and why? (Hint: use the fact that I is not principal).
- Is a submodule of a finitely generated free module necessarily free?
- Find any generating set for I as a $\mathbb{Z}$-module. Show that the generating set consisting of only 2 and $1 + \sqrt{-5}$ generates I as a $\mathbb{Z}$-module. Is I a free Z-module?

(2) (4 marks) Let R be a commutative ring and M be an R-module. Show the following conditions are equivalent.

- M is a Noetherian R-module.
- M satisfies the ascending chain condition.
- Every nonempty set of submodules of M contains some maximal element.